SVR TSTM 2025 : QG motivation

Recall: $\frac{d\vec{u}}{dt} = \underbrace{-f\hat{k}\times\vec{v}}_{①} \underbrace{- \vec{\nabla}_p\Phi}_{②}$ ① Coriolis ② PGF } when in balance, $\frac{d\vec{u}}{dt} = 0$

$\boxed{\text{Geostrophy}}$

What about when $\frac{d\vec{u}}{dt} \neq 0$? $\boxed{\text{Quasi-geostrophy}}$ Yay!

QG ω derivation

Start w/ QG ζ and T equations:

① QGζ: $\frac{\partial \zeta_g}{\partial t} = \underbrace{-\vec{v}_g \cdot \vec{\nabla}_p \zeta_g}_{①} \underbrace{-\beta v_g}_{②} \underbrace{-\delta f_0}_{③} \underbrace{-K\zeta_g}_{④}$

① "horizontal" advection of ζ ($\zeta_g + f$)
② same for f
③ stretching of f
④ frictional effects (small. $\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} = -K\zeta_g$)

② QGT: $\frac{\partial T}{\partial t} = \underbrace{-\vec{v}_g \cdot \vec{\nabla}_p T}_{①} + \underbrace{\omega\sigma\frac{p}{R}}_{②} + \underbrace{\frac{1}{c_p}\frac{dQ}{dt}}_{③}$

① "horizontal" advection
② vertical motion
③ diabatic effects

$\sigma \doteq -\frac{RT}{p}\frac{\partial \ln\theta}{\partial p}$ "static stability parameter"

Now, assume:

→ Adiabatic ($Q = 0$)
→ Frictionless ($K = 0$)
→ Constant f plane ($\beta = 0$)

This simplifies ① & ② too...

total 1

$$① \quad \frac{\partial \zeta_g}{\partial t} = -\vec{v}_g \cdot \vec{\nabla}_p \zeta_g - \delta f_0 \quad \Longrightarrow \quad \frac{1}{f_0}\vec{\nabla}^2 \frac{\partial \Phi}{\partial t} = -\vec{v}_g \cdot \vec{\nabla}_p\left(\frac{1}{f_0}\vec{\nabla}^2\Phi + f\right) + f_0 \frac{\partial \omega}{\partial p}$$

$$② \quad \frac{\partial T}{\partial t} = -\vec{v}_g \cdot \vec{\nabla}_p T + \omega \sigma \frac{p}{R}$$

We can express ζ_g in terms of χ ...

$$\zeta_g \doteq \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial}{\partial x}\left(\frac{1}{f_0}\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f_0}\frac{\partial \Phi}{\partial y}\right)$$

$$= \frac{1}{f_0}\left[\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right] = \frac{1}{f_0}\vec{\nabla}^2 \Phi$$

$\left(\text{and } \delta = -\frac{\partial \omega}{\partial p}\right)$

So now we have:

$$① \quad \frac{1}{f_0}\vec{\nabla}^2 \chi = -\vec{v}_g \cdot \vec{\nabla}_p\left(\frac{1}{f_0}\vec{\nabla}^2\Phi + f\right) + f_0\frac{\partial \omega}{\partial p}$$

$$② \quad -\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right) = \vec{v}_g \cdot \vec{\nabla}\left(\frac{\partial \Phi}{\partial p}\right) + \sigma\omega \quad \left(\text{sub } T = -\frac{\partial \Phi}{\partial p}\right) \text{ "thickness"}$$

Step 1: Take $f_0 \frac{\partial}{\partial p}(①)$

$$f_0 \frac{\partial}{\partial p}\left[\frac{1}{f_0}\vec{\nabla}^2 \chi\right] = f_0\frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \vec{\nabla}_p\left(\frac{1}{f_0}\vec{\nabla}^2\Phi + f\right)\right] + f_0\frac{\partial}{\partial p}\left[f_0 \frac{\partial \omega}{\partial p}\right]$$

$\chi = \frac{\partial \Phi}{\partial t}$

$$\boxed{ⓐ \quad \frac{\partial}{\partial p}\left(\vec{\nabla}^2 \frac{\partial \Phi}{\partial t}\right) = f_0\frac{\partial}{\partial p}\left(-\vec{v}_g \cdot \vec{\nabla}_p(\zeta_g + f)\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2}}$$

Step 2: Take $\vec{\nabla}^2(②)$

$$\boxed{ⓑ \quad \vec{\nabla}^2\left(-\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right)\right) = \vec{\nabla}^2\left(\vec{v}_g \cdot \vec{\nabla}\frac{\partial \Phi}{\partial p}\right) + \sigma\vec{\nabla}^2\omega}$$

Step 3: ⓐ + ⓑ (LHS = 0)

$$0 = f_0 \frac{\partial}{\partial p}\left(-\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f)\right) + f_0^2 \frac{\partial^2 \omega}{\partial p^2} + \vec{\nabla}^2\left(\vec{V}_g \cdot \vec{\nabla} \frac{\partial \Phi}{\partial p}\right) + \sigma \vec{\nabla}^2 \omega$$

Rearrange...

$$-\sigma \vec{\nabla}^2 \omega - f_0^2 \frac{\partial^2 \omega}{\partial p^2} = f_0 \frac{\partial}{\partial p}\left(-\vec{V}_g \cdot \vec{\nabla}(\zeta_g + f)\right) + \vec{\nabla}^2\left(\vec{V}_g \cdot \vec{\nabla}\left(\frac{\partial \Phi}{\partial p}\right)\right)$$

Multiply by -1 and combine LHS:

$$\boxed{\underbrace{\left(\sigma \vec{\nabla}^2 + f_0^2 \frac{\partial^2}{\partial p^2}\right)}_{1} \omega = - f_0 \frac{\partial}{\partial p}\left(-\vec{V}_g \cdot \underbrace{\vec{\nabla}(\zeta_g + f)}_{2}\right) - \vec{\nabla}^2\left(-\vec{V}_g \cdot \underbrace{\vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)}_{3}\right)}$$

1. $\sim -\omega$ $(w > 0)$
2. Differential vorticity advection
3. Temperature advection

Key takeaways:

1. For lower stability ($\sigma \downarrow$) and equal forcing (RHS), ω increases
2. Increasing CVA w/ height $\Rightarrow$ rising motion
   $\frac{\partial}{\partial z}(-\vec{V}_g \cdot \vec{\nabla}\zeta) > 0 \Rightarrow -\frac{\partial}{\partial p}(-\vec{V}_g \cdot \vec{\nabla}\zeta) > 0 \Rightarrow \boxed{-\omega} \Rightarrow \boxed{w > 0}$
3. WAA $\rightarrow$ rising motion
   $-\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right) > 0 \Rightarrow \vec{\nabla}^2(\ldots) < 0 \Rightarrow -\vec{\nabla}^2(\ldots) > 0 \Rightarrow \boxed{\omega < 0} \Rightarrow \boxed{w > 0}$

$\boxed{\text{END}}$

See "Lecture 2-ish" for examples

QG χ derivation:

Back to our starting equations...

① $$\frac{1}{f_0}\vec{\nabla}_p^2\chi = -\vec{V}_g\cdot\vec{\nabla}_p\left(\frac{1}{f_0}\vec{\nabla}^2\phi + f\right) + f_0\frac{\partial\omega}{\partial p}$$

② $$-\frac{\partial}{\partial p}\chi = \vec{V}_g\cdot\vec{\nabla}_p\left(\frac{\partial\phi}{\partial p}\right) + \sigma\omega$$

Step 1: Take f_0 (①)

ⓐ $$\vec{\nabla}_p^2\chi = f_0\left(-\vec{V}_g\cdot\vec{\nabla}_p\left(\frac{1}{f_0}\vec{\nabla}^2\phi + f\right)\right) + f_0^2\frac{\partial\omega}{\partial p}$$

Step 2: Take $-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}$(②)

$$-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\frac{\partial\chi}{\partial p}\right) = -\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(\vec{V}_g\cdot\vec{\nabla}\left(\frac{\partial\phi}{\partial p}\right)\right) - \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}(\sigma\omega)$$

assume $\frac{\partial\sigma}{\partial p} = 0$ (!!)

ⓑ $$\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\chi = \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\vec{V}_g\cdot\vec{\nabla}\left(\frac{\partial\phi}{\partial p}\right)\right) - f_0^2\frac{\partial\omega}{\partial p}$$

Step 3: ⓐ+ⓑ (last term cancels)

$$\boxed{\underbrace{\left(\vec{\nabla}_p^2 + \frac{\partial^2}{\partial p^2}\right)\chi}_{①} = \underbrace{f_0\left(-\vec{V}_g\cdot\vec{\nabla}_p(\zeta_g + f)\right)}_{②} - \frac{f_0^2}{\sigma}\underbrace{\frac{\partial}{\partial p}\left(-\vec{V}_g\cdot\vec{\nabla}\left(-\frac{\partial\phi}{\partial p}\right)\right)}_{③}}$$

① $\sim -\chi$ (height falls)

② Vorticity advection

③ Differential temperature advection

Key takeaways:

① Not dependent on static stability

② CVA ⇒ height falls

$$-\vec{V}_g \cdot \vec{\nabla}\zeta > 0 \Rightarrow \boxed{\chi < 0}$$

③ DCAA ⇒ height falls

$$\frac{\partial}{\partial z}\left[-\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] < 0 \Rightarrow -\frac{\partial}{\partial p}(\ldots) > 0 \Rightarrow \boxed{\chi < 0}$$

$\boxed{\text{END}}$

See "lecture 2-ish" for examples